Number Theoretic Functional Equations

James Rickards
jamesarickards@hotmail.com

1 Introduction

Number theoretic functional equations combine the algebraic techniques of functional equations with various number theoretic techniques. Things to keep in mind:

- Standard functional equation tricks still apply. For example, trying to find special values ($f(0), f(1)$, etc.), proving injectivity/surjectivity, canceling terms, etc.
- If you can reduce the functional equation to Cauchy's functional equation and the range of your function is $\mathbb{Q}$, then you are done
- For problems with divisibility, try to get an equation which divides a prime or a power of a prime
- A common method with functional inequalities is showing $f(x) \leq x$ and $f(x) \geq x$; with number theoretic functionals you can often do the analogous thing with $f(x) \mid x$ and $x \mid f(x)$.
- If you are trying to construct a function, it will often be possible to do so with induction/recursion as your range will often be a countable set (like $\mathbb{Z}^+$).

2 Example

Example 1 (IMOSL 2004 N3). *Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying*

$$\left(f(m)^2 + f(n)\right) \mid \left(m^2 + n\right)^2$$

for all positive integers m, n.

Solution. First try $m = n = 1$ and we derive $f(1)^2 + f(1) \mid 4$. Note that if $f(1) \geq 2$ then $4 \geq f(1)^2 + f(1) \geq 6$, which is a contradiction. Thus $f(1) = 1$. Plug in $m = 1$ and $n = 1$, and we derive

$$f(m)^2 + 1 \mid (m^2 + 1)^2;$$
$$f(n) + 1 \mid (n + 1)^2.$$

Motivated by this, consider $n = p-1$ where p is a prime. This implies that $f(p-1)+1 \in \{1, p, p^2\}$. Since the range of f is the positive integers, we must have $f(p-1) \in \{p-1, p^2-1\}$. Assume that $f(p-1) = p^2-1$ for some prime p. Taking $m = p-1$ and $n = 1$ we derive $(p^2-1)^2+1 \mid ((p-1)^2+1)^2$, whence $(p^2-1)^2+1 \leq ((p-1)^2+1)^2$. Expanding gives $p^4-2p^2+2 \leq p^4-4p^3+8p^2-8p+4$, and so $2p^3-5p^2+4p-1 \leq 0$. This is false for $p=2$, and for $p \geq 3$ we have

$$0 \geq 2p^3-5p^2+4p-1 = p^2(2p-5)+4p-1 > 0,$$

contradiction.

Therefore $f(p-1)=p-1$ for all primes p. How can we boost this up to $f(n)=n$ for all n? Take $m=p-1$, and then we have

$$(f(n)+(p-1)^2) \mid (n+(p-1)^2)^2.$$

This implies $(f(n)+(p-1)^2) \mid (n-f(n))^2$ for all n and all primes p. If $f(n) \neq n$, then for sufficiently large p this cannot hold, contradiction. Thus $f(n)=n$ for all n. $\square$

3 Problems

I will adopt the David Arthur style of ordering problems into 3 sections: A,B,C. A-level problems should be approximately CMO level, B level problems would be easy-medium IMO problems, and C level would be medium-hard IMO or beyond problems. This ordering is of course somewhat subjective, so don't be surprised if you find some problems to be out of place.

A1 Prove there do not exist any functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all $x \in \mathbb{Z}^+$ we have

$$f(2f(x)) = x + 1998.$$

A2 Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all x, y in $\mathbb{Q}^+$.

A3 Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$, such that $f(x+1) = f(x)+1$ and $f(x^3) = f(x)^3$ for all x.

A4 Does there exist a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying $f(f(n-1)) = f(n+1) - f(n)$ for all integers $n \geq 2$?

B1 Let $a, b \in \mathbb{N}$ with $1 \leq a \leq b$, and $M = \left[\frac{a+b}{2}\right]$. Define a function $f : \mathbb{Z} \mapsto \mathbb{Z}$ by

$$f(n) = \begin{cases} n+a, & \text{if } n \leq M, \\ n-b, & \text{if } n > M. \end{cases}$$

Let $f^1(n) = f(n)$, $f_{i+1}(n) = f(f^i(n))$, $i = 1, 2, \ldots$ Find the smallest natural number k such that $f^k(0) = 0$.

B2 Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m, n.

B3 Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b+f(a)-1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

B4 Find all surjective functions $f:\mathbb{Z}^+\to\mathbb{Z}^+$ such that for all positive integers m,n we have

$$m \mid n \Leftrightarrow f(m) \mid f(n)$$

B5 Let r and s be two rational numbers. Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that for all $x,y\in\mathbb{Q}$ we have

$$f(x+f(y)) = f(x+r)+y+s.$$

B6 Let $f:\mathbb{Q}\to\mathbb{Q}$ be a function such that for any $x,y\in\mathbb{Q}$, the number $f(x+y)-f(x)-f(y)$ is an integer. Decide whether it follows that there exists a constant c such that $f(x)-cx$ is an integer for every rational number x.

B7 Let $n\geq 1$ be an odd integer. Determine all functions f from the set of integers to itself, such that for all integers x and y the difference $f(x)-f(y)$ divides x^n-y^n.

B8 Let f be a non-constant function from the set of positive integers into the set of positive integer, such that $a-b$ divides $f(a)-f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

B9 Let $f:\mathbb{Z}\to\mathbb{Z}^+$ be a function. Suppose that for any two integers m,n, we have $f(m-n) \mid f(m)-f(n)$. Prove that if m,n are integers with $f(m)\leq f(n)$, then $f(m)\mid f(n)$.

B10 Determine all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all $x,y\in\mathbb{Z}$ we have

$$f(x-f(y)) = f(f(x))-f(y)-1.$$

B11 Find all integer polynomials f such that there are infinitely many pairs of relatively prime natural numbers (a,b) so that $a+b \mid f(a)+f(b)$.

B12 Does there exist a function $f:\mathbb{Q}\to\{-1,1\}$ such that if x,y are distinct rational numbers with $xy=1$ or $x+y\in\{0,1\}$, then $f(x)f(y)=-1$?

B13 Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all integers a,b,c with $a+b+c=0$ we have

$$f(a)^2+f(b)^2+f(c)^2 = 2f(a)f(b)+2f(b)f(c)+2f(c)f(a).$$

B14 Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x)=x$ is equal to $P(n)$ for every $n\geq 1$, where T^n denotes the n-fold application of T.

B15 Find all surjective functions $f:\mathbb{N}\to\mathbb{N}$ such that for every $m,n\in\mathbb{N}$ and every prime p, the number $f(m+n)$ is divisible by p if and only if $f(m)+f(n)$ is divisible by p.

C1 For every $n \in \mathbb{Z}^+$ let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ with the following properties:

- $d\left(f(x)\right) = x$ for all $x \in \mathbb{N}$.
- $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all $x,\ y \in \mathbb{N}$.

C2 Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$\left(g(m)+n\right)\left(g(n)+m\right)$$

is a perfect square for all $m, n \in \mathbb{N}$.

C3 Let k be a positive integer and $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ a function. We call f $k-$good if we have

$$\gcd(f(m)+n, f(n)+m) \leq k$$

for all distinct positive integers m, n. Prove that there does not exist a $1-$good function, but there does exist a $2-$good function.

C4 Let n be a given integer. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$, such that for all integers x, y we have

$$f(x+y+f(y)) = f(x) + ny.$$

C5 Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x)f(y)f(x+y) = f(xy)(f(x)+f(y))$ for all $x, y \in \mathbb{Q}$.

C6 Find all $f : \mathbb{Z} \to \mathbb{R}$ satisfying

$$f(x+y) + f(1) + f(xy) = f(x) + f(y) + f(1+xy)$$

4 Hints

A1 Note that f is injective and $2f(x)$ is even.

A2 Construct a function satisfying $f(xy) = f(x)f(y)$ and $f(f(x)) = \frac{1}{x}$.

A3 The only solution is $f(x) = x$.

A4 Show that f is increasing and reach a contradiction.

B1 The answer is $\frac{a+b}{\gcd(a,b)}$; try to prove that it is at least this number, and at most this number.

B2 Show that $f(1) = 1$ and try to show that $f(n) \leq n$ and $f(n) \geq n$.

B3 Show that $f(1) = 1$ and $f(n) = (n-1)f(2) - (n-2)$.

B4 The solution set is to take any bijection of the primes to themselves, and use this to permute the primes in the prime factorization of n.

B5 Try reducing it to something like Cauchy's functional equation.

B6 The answer is no, there exists a counterexample!

B7 Note that $f(p) - f(0) \mid p^n$, so $f(p) = f(0) \pm p^d$ for $d \leq n$. Work with this.

B8 Try to emulate the proof of there being infinitely many primes dividing a non-constant polynomial with integer coefficients.

B9 Show that $f(n) \mid f(0)$ for all n and that $f(-n) = f(n)$. Now play around with $m, n, m-n, m+n$.

B10 Show that $-1 \in \operatorname{Im}(f)$, and use this to eventually show that $f(-1) + 1 = f(x+1) - f(x)$, implying that f is linear.

B11 Split f as a sum of even and odd polynomials.

B12 Yes there does; consider the equalities $s(x) = s(x+1)$ and $s(x) = -s(1/x)$; does this remind you of something?

B13 Do not attempt this problem, it is a terrible problem. But seriously, this question is a good test of casework and exotic solutions. There are three families of solutions, with one having $f(x)$ having period 2, and another with $f(x)$ having period 4.

B14 Define a_i to be the number of integers n for which $T^i(n) = n$, and i is the smallest satisfying this. Work with this.

B15 Start by proving that f must be bijective.

C1 Show that $f(1) = 1$, $f(p^a) = p^{p^a - 1}$ for primes p, and show that $f(xy) = f(x)f(y)$ if x, y are relatively prime.

C2 Show that $g(n) \neq g(n+1), g(n+2)$ and that $g(n+1) = g(n) \pm 1$.

C3 To disprove the existence of a 1-good function, look modulo 2. For a 2-good function, try to construct it by induction; something should go a little wrong. Try to modify your induction appropriately (there is an alternate explicit construction).

C4 Define $A = \{y : f(x+y) = f(x) \text{ for all } x\}$, and prove that $A = m\mathbb{Z}$ for some integer m. The cases $n < 0$, $n = 0$, $n > 0$ will be slightly different.

C5 There are seven distinct solutions. The three "easier" solutions are $f(x) = 0, 2, x$. The tricky solution is

$$g\left(\frac{p}{q}\right) = \begin{cases} 0 & \text{if } pq \equiv 0 \pmod 3 \\ -2 & \text{if } pq \not\equiv p - q \equiv 0 \pmod 3 \\ 2 & \text{if } pq \not\equiv p + q \equiv 0 \pmod 3 \end{cases}$$

The other three solutions are the above for $x > 0$ combined with $f(x) = 0$ for $x \leq 0$.

C6 Work recursively to show that if you know $f(0), f(1), f(2), f(3), f(4)$, then you can deduce $f(n)$ for all integers n.